

A graph-theoretic condition for irreducibility of a set of cone preserving matrices

Murad Banaji^{a,*}, Andrew Burbanks^a

^a*Department of Mathematics, University of Portsmouth, Lion Gate Building, Lion Terrace, Portsmouth, Hampshire PO1 3HF, UK.*

Abstract

Given a closed, convex and pointed cone K in \mathbb{R}^n , we present a result which infers K -irreducibility of sets of K -quasipositive matrices from strong connectedness of certain bipartite digraphs. The matrix-sets are defined via products, and the main result is relevant to applications in biology and chemistry. Several examples are presented.

Keywords: partial order, convex cone, irreducibility, strong monotonicity
2000 MSC: 15B48, 15B35

1. Introduction

A digraph G is strongly connected, or irreducible, if given any vertices u and v , there exists a (directed) path from u to v in G . It is well known that a digraph is strongly connected if and only if its adjacency matrix is irreducible [1]. Here, given a cone K , we present a result which infers K -irreducibility of sets of K -quasipositive matrices from strong connectedness of associated bipartite digraphs. Graph-theoretic approaches to K -irreducibility of sets of K -positive matrices have been described in earlier work [2, 3, 4]. These approaches are somewhat different in structure and philosophy to that described here. We will comment further on this in the concluding section.

We will be interested in closed, convex cones in \mathbb{R}^n which are additionally pointed (i.e., if $y \in K$ and $y \neq 0$, then $-y \notin K$). Closed, convex and pointed cones will be abbreviated as **CCP** cones. We do not assume the cones are solid (i.e., have nonempty interior in \mathbb{R}^n) – however if a CCP cone is, additionally, solid, then it will be termed a **proper** cone. For basic definitions and results on cones in \mathbb{R}^n the reader is referred to [1, 5].

Let $K \subset \mathbb{R}^n$ be a CCP cone. Consider an $n \times n$ matrix M . Recall that M is **K -positive** if $MK \subseteq K$. Defining $\mathbb{R}_{\geq 0}^n$ to be the (closed) nonnegative orthant in \mathbb{R}^n , a nonnegative matrix is then $\mathbb{R}_{\geq 0}^n$ -positive. We will refer to M as **K -quasipositive** if there exists an $\alpha \in \mathbb{R}$ such that $M + \alpha I$ is K -positive.

*Corresponding author: murad.banaji@port.ac.uk.

$\mathbb{R}_{\geq 0}^n$ -quasipositive matrices – generally referred to simply as quasipositive, or Metzler – are just those with nonnegative off-diagonal entries. We define M to be **K -reducible** if there exists a nontrivial face F of K such that M leaves $\text{span } F$ invariant. This is a slight generalisation of the original definition of K -reducibility for K -positive matrices [6] in order to allow us to apply the term to matrices which are not necessarily K -positive. A matrix which is not K -reducible is **K -irreducible**. Note that an irreducible matrix could be termed $\mathbb{R}_{\geq 0}^n$ -irreducible in this terminology. Alternatively any other orthant in \mathbb{R}^n could be chosen as K .

Remark 1. *Given a CCP cone $K \subset \mathbb{R}^n$, an $n \times n$ matrix M is K -irreducible if and only if $M + \alpha I$ is K -irreducible for each $\alpha \in \mathbb{R}$. In one direction we choose $\alpha = 0$. The other direction follows because given any face F of K , $\text{span } F$ is a vector subspace of \mathbb{R}^n .*

Motivation from dynamical systems. Motivation for examining K -irreducibility of a set of K -quasipositive matrices comes from the theory of monotone dynamical systems [7, 8]. Convex, pointed cones define partial orders in a natural way. Given a proper cone K and a C^1 vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, if the Jacobian matrix $Df(x)$ is K -quasipositive and K -irreducible at each $x \in \mathbb{R}^n$, then the associated local flow is strongly monotone with respect to the partial order defined by K . This result and a variety of technical modifications provide useful conditions which can be used to deduce the asymptotic behaviour of dynamical systems.

Remark 2. *Although the main results on monotone dynamical systems require the order cone to be solid, a CCP cone K which fails to be solid is as useful as a proper one when the local flow or semiflow leaves cosets of $\text{span } K$ invariant. Trivially, K has nonempty relative interior in $\text{span } K$, and attention can be restricted to cosets of $\text{span } K$. This situation arises frequently in applications to biology and chemistry.*

2. Some background material

Sets of matrices. It is convenient to use some notions from qualitative matrix theory. Let M be a real matrix.

1. $\mathcal{Q}(M)$, the qualitative class of M [9] is the set of all matrices with the same dimensions and sign pattern as M , namely if $N \in \mathcal{Q}(M)$, then $M_{ij} > 0 \Rightarrow N_{ij} > 0$, $M_{ij} < 0 \Rightarrow N_{ij} < 0$ and $M_{ij} = 0 \Rightarrow N_{ij} = 0$.
2. $\mathcal{Q}_0(M)$ will stand for the closure of $\mathcal{Q}(M)$, namely if $N \in \mathcal{Q}_0(M)$, then $M_{ij} > 0 \Rightarrow N_{ij} \geq 0$, $M_{ij} < 0 \Rightarrow N_{ij} \leq 0$ and $M_{ij} = 0 \Rightarrow N_{ij} = 0$.
3. $\mathcal{Q}_1(M)$ will be the set of all matrices N with the same dimensions as M and satisfying $M_{ij}N_{ij} \geq 0$.

Clearly, $\mathcal{Q}(M) \subseteq \mathcal{Q}_0(M) \subseteq \mathcal{Q}_1(M)$.

Remark 3. Suppose K is an orthant in \mathbb{R}^n and M is an $n \times n$ K -quasipositive matrix. It can easily be shown that each matrix in $\mathcal{Q}_0(M)$ is K -quasipositive. Since in this case K -irreducibility is simply irreducibility, if M is K -irreducible then each matrix in $\mathcal{Q}(M)$ is K -irreducible. However, the same is not true for $\mathcal{Q}_0(M)$ which, after all, contains the zero matrix.

Notation for matrices. Given any matrix M , we refer to the k th column of M as M_k and the k th row of M as M^k . We define the new matrix $M^{(k)}$ by $M_{ij}^{(k)} = M_{ij}$ if $i = k$ and $M_{ij}^{(k)} = 0$ otherwise: i.e., $M^{(k)}$ is derived from M by replacing all entries not in the k th row with zeros. We define a set of matrices \mathbf{M} to be **row-complete** if $M \in \mathbf{M} \Rightarrow M^{(k)} \in \mathbf{M}$ for each k . Clearly, given any matrix M , $\mathcal{Q}_0(M)$ is row-complete; but much smaller sets can be row-complete. For example, given some fixed $m \times n$ matrix N ,

$$\mathbf{M} = \{DN : D \text{ is a nonnegative } m \times m \text{ diagonal matrix}\}$$

is row-complete.

Digraphs associated with square matrices. Given an $n \times n$ matrix M , let G_M be the associated digraph on n vertices u_1, \dots, u_n defined in the usual way: the arc $u_i u_j$ exists in G_M iff $M_{ij} \neq 0$.

Remark 4. Following on from Remark 3, if K is an orthant in \mathbb{R}^n , then K -irreducibility of an $n \times n$ matrix M is equivalent to strong connectedness of G_M .

Bipartite digraphs associated with matrix-pairs. Given an $n \times m$ matrix A and an $m \times n$ matrix B , define a bipartite digraph $G_{A,B}$ on $n + m$ vertices as follows: associate a set of n vertices u_1, \dots, u_n with the rows of A (we will refer to these as the “row vertices” of $G_{A,B}$); associate another m vertices v_1, \dots, v_m with the columns of A (we will refer to these as the “column vertices” of $G_{A,B}$); add the arc $u_i v_j$ iff $A_{ij} \neq 0$; add the arc $v_j u_i$ iff $B_{ji} \neq 0$.

Remark 5. This is a specialisation of the general construction of block-circulant digraphs from sets of appropriately dimensioned matrices in [10]. If A and B are binary matrices (with AB a square matrix), then the adjacency matrix of $G_{A,B}$ is simply

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

The context of the main result. Fundamental early results on convergence in monotone dynamical systems [11] apply to systems with Jacobian matrices which are quasipositive and irreducible (in our terminology $\mathbb{R}_{\geq 0}^n$ -quasipositive and $\mathbb{R}_{\geq 0}^n$ -irreducible). Generalising from $\mathbb{R}_{\geq 0}^n$ to all orthants is straightforward: where K is an orthant, there is a simple graph-theoretic test [7] to decide K -quasipositivity of a given $n \times n$ matrix M . By Remarks 3 and 4, K -quasipositivity extends to all of $\mathcal{Q}_0(M)$, and K -irreducibility of some $M' \in \mathcal{Q}_0(M)$ is equivalent to strong connectedness of $G_{M'}$.

We are interested in how this situation generalises when K is not necessarily an orthant. In general, given a K -quasipositive matrix M , we can rarely expect all matrices in $\mathcal{Q}_0(M)$ to be K -quasipositive. However the following situation is not uncommon: there are matrices A and \tilde{B} such that AB is K -quasipositive for each $B \in \mathcal{Q}_0(\tilde{B})$. The practical relevance is to applications in biology and chemistry where Jacobian matrices often have a constant initial factor, but a second factor with variable entries whose signs are, however, known. A number of particular examples were presented in [12]. More generally it may also happen that AB is K -quasipositive for each B in some set \mathbf{B} , where \mathbf{B} is a proper subset of the closure of some qualitative class.

Given a set of K -quasipositive matrices of the form $\{AB : B \in \mathbf{B}\}$, we would hope that there is a natural graph-theoretic test to decide which members of this set are also K -irreducible. Theorem 1 provides precisely such a condition: provided \mathbf{B} is row-complete and the initial factor A satisfies a mild genericity condition, K -irreducibility of AB follows from strong connectedness of the bipartite digraph $G_{A,B}$. In the special case where K is the nonnegative orthant, A is the identity matrix, and \mathbf{B} is the set of nonnegative matrices, the results reduce to well known ones.

Remark 6. *Our motivation for considering row-complete sets of matrices is as follows: consider a set of K -quasipositive matrices of the form $\{AB : B \in \mathbf{B}\}$, where \mathbf{B} is row-complete. Clearly*

$$AB = A \sum_k B^{(k)} = \sum_k AB^{(k)} = \sum_k A_k B^k.$$

So any matrix in the set can be written as a sum of rank 1 K -quasipositive matrices.

3. The main result

From now on $K \subset \mathbb{R}^n$ will be a CCP cone in \mathbb{R}^n , A an $n \times m$ matrix and \mathbf{B} a row-complete set of $m \times n$ matrices. For any $B \in \mathbf{B}$, AB is an $n \times n$ matrix. The main result of this paper is:

Theorem 1. *Assume that $\text{Im } A \not\subseteq \text{span } F$ for any nontrivial face F of K . Suppose that for each $B \in \mathbf{B}$, AB is K -quasipositive. Then whenever $G_{A,B}$ is strongly connected, AB is also K -irreducible.*

An immediate corollary of Theorem 1 is:

Corollary 2. *Assume that $\text{Im } A \not\subseteq \text{span } F$ for any nontrivial face F of K . Suppose that for each $B \in \mathbf{B}$, AB is K -positive. Then whenever $G_{A,B}$ is strongly connected, AB is also K -irreducible.*

PROOF. K -positivity of AB implies K -quasipositivity of AB . The result now follows from Theorem 1. \square

Remark 7. Note that if $\text{Im } A \subseteq \text{span } F$ for some nontrivial face F of K , then trivially AB is K -reducible. To see that the assumption that $\text{Im } A \not\subseteq \text{span } F$ is in general necessary in Theorem 1, consider the matrices

$$\Lambda = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \geq 0$. Let $K = \{\Lambda z : z \in \mathbb{R}_{\geq 0}^2\}$. Then

$$AB\Lambda_1 = (a + c + 2(b + d))\Lambda_1, \quad AB\Lambda_2 = (2(a + c) + b + d)\Lambda_1$$

which are both clearly in K for any $a, b, c, d \geq 0$. So AB is K -positive. On the other hand $F = \{r\Lambda_1 : r \geq 0\}$ satisfies $(AB)F \subseteq F$ for all B , so AB is K -reducible for all nonnegative values of a, b, c, d . However, for $a, b, c, d > 0$, $G_{A,B}$ is the complete bipartite digraph $K_{2,2}$, which is obviously strongly connected.

4. Proofs

We need some preliminary lemmas in order to prove Theorem 1. The following is proved as Lemma 4.4 in [12]:

Lemma 3. Let F be a face of K , $v_1, v_2 \in F$, and $w \in \mathbb{R}^n$. If there exist $\alpha, \beta > 0$ such that $v_1 + \alpha w \in K$ and $v_2 - \beta w \in K$, then $w \in \text{span}(F)$.

PROOF. Define $y_1 \equiv v_1 + \alpha w$ and $y_2 \equiv v_2 - \beta w$. Then $y_3 \equiv y_1 + (\alpha/\beta)y_2 = v_1 + (\alpha/\beta)v_2 \in F$. Since $y_3 \in F$, $y_3 - y_1 = (\alpha/\beta)y_2 \in K$, and $y_1 \in K$, by the definition of a face, $y_1 \in F$. So $w = (y_1 - v_1)/\alpha \in \text{span}(F)$. \square

Extremals. A one dimensional face of K will be termed an extremal ray or an extremal for short, and any nonzero vector in an extremal ray will be an extremal vector of K .

Lemma 4. Suppose that for each $B \in \mathbf{B}$, AB is K -quasipositive. Let \mathcal{E} be an extremal vector of K . Then for each (fixed) j either $A_j = r\mathcal{E}$ for some real number r or $B^j\mathcal{E} \geq 0$ for all $B \in \mathbf{B}$ or $B^j\mathcal{E} \leq 0$ for all $B \in \mathbf{B}$.

PROOF. Suppose there exist j and $\overline{B}, \underline{B} \in \mathbf{B}$ such that $p_1 \equiv \overline{B}^j\mathcal{E} > 0$ and $p_2 \equiv -\underline{B}^j\mathcal{E} > 0$. Note that $A\overline{B}^{(j)} = A_j\overline{B}^j$ and $A\underline{B}^{(j)} = A_j\underline{B}^j$. Since \mathbf{B} is row-complete, $\overline{B}^{(j)}, \underline{B}^{(j)} \in \mathbf{B}$ and so $A\overline{B}^{(j)}, A\underline{B}^{(j)}$ are K -quasipositive. Let α_1 and α_2 be such that $A\overline{B}^{(j)}\mathcal{E} + \alpha_1\mathcal{E} \in K$ and $A\underline{B}^{(j)}\mathcal{E} + \alpha_2\mathcal{E} \in K$ respectively. We can assume (w.l.o.g.) that $\alpha_1, \alpha_2 > 0$. Then:

$$A\overline{B}^{(j)}\mathcal{E} + \alpha_1\mathcal{E} = A_j\overline{B}^j\mathcal{E} + \alpha_1\mathcal{E} = \alpha_1\mathcal{E} + p_1A_j \in K$$

and

$$A\underline{B}^{(j)}\mathcal{E} + \alpha_2\mathcal{E} = A_j\underline{B}^j\mathcal{E} + \alpha_2\mathcal{E} = \alpha_2\mathcal{E} - p_2A_j \in K.$$

Now, by Lemma 3, these two equations imply that $A_j = r\mathcal{E}$ for some r . \square

Remark 8. An alternative phrasing of Lemma 4 is that if A_j is not collinear with \mathcal{E} then either $B^j \in \mathcal{Q}_1(\mathcal{E}^T)$ for all $B \in \mathbf{B}$ or $B^j \in \mathcal{Q}_1(-\mathcal{E}^T)$ for all $B \in \mathbf{B}$.

Lemma 5. Let F be a nontrivial face of K , $\{J(k)\}$ be a finite set of $n \times n$ K -quasipositive matrices and $J = \sum J(k)$. If there exists $x \in F$ such that $Jx \in \text{span } F$, then $J(k)x \in \text{span } F$ for each k .

PROOF. By K -quasipositivity of each $J(k)$ we can write

$$J(k)x = p_k + q_k$$

where $p_k \in (K \setminus F) \cup \{0\}$ and $q_k \in \text{span } F$. Summing, we get

$$Jx = p + q$$

where $q = \sum_k q_k \in \text{span } F$ and $p = \sum_k p_k \in (K \setminus F) \cup \{0\}$. Now if $Jx \in \text{span } F$ then $p = 0$. Since $p_k \in K$ and K is pointed, this implies that $p_k = 0$ for each k . So $J(k)x = q_k \in \text{span } F$ for each k , proving the lemma. \square

Lemma 6. Let AB be K -quasipositive for each $B \in \mathbf{B}$ and let F be a nontrivial face of K spanned by (pairwise independent) extremal vectors $\{\Lambda_i\}_{i \in \mathcal{I}}$. Choose and fix some $B \in \mathbf{B}$. Then given any nonempty set $\mathcal{R} \subseteq \{1, \dots, m\}$, either (i) $A_k \in \text{span } F$ for some $k \in \mathcal{R}$ or (ii) $B^k \Lambda_i = 0$ for each $k \in \mathcal{R}, i \in \mathcal{I}$, or (iii) there exists $\Lambda_i \in F$ such that $\sum_{k \in \mathcal{R}} A_k B^k \Lambda_i \notin \text{span } F$.

PROOF. For each k , recall that $B^{(k)} \in \mathbf{B}$, so $AB^{(k)} = A_k B^k$ is K -quasipositive. Suppose (iii) does not hold, i.e., $\sum_{k \in \mathcal{R}} A_k B^k \Lambda_i \in \text{span } F$ for each $\Lambda_i \in F$. Applying Lemma 5 with $J(k) = A_k B^k$, we get for each $k \in \mathcal{R}$ and each $\Lambda_i \in F$ that $A_k B^k \Lambda_i \in \text{span } F$. So for each fixed $k \in \mathcal{R}$, either $A_k \in \text{span } F$ or $B^k \Lambda_i = 0$ for all $i \in \mathcal{I}$. \square

PROOF OF THEOREM 1. We show that if AB is K -reducible for some $B \in \mathbf{B}$, then $G_{A,B}$ cannot be strongly connected. Let $\{\Lambda_i\}$ be a set of pairwise independent extremal vectors of K . Let F be a nontrivial face of K such that $\text{span } F$ is left invariant by AB , and let \mathcal{I} be the indices of vectors Λ_i in F , i.e., $i \in \mathcal{I} \Leftrightarrow \Lambda_i \in F$. Let $\mathcal{R} \subset \{1, \dots, m\}$ be defined by $k \in \mathcal{R} \Leftrightarrow A_k \in \text{span } F$. \mathcal{R} may be empty, but by assumption cannot be all of $\{1, \dots, m\}$ since $\text{Im } A \not\subseteq \text{span } F$. So \mathcal{R}^c , the complement of \mathcal{R} , is nonempty.

Choose any $x \in F$. We have

$$ABx = \sum_k A_k B^k x = \sum_{k \in \mathcal{R}} A_k B^k x + \sum_{k \in \mathcal{R}^c} A_k B^k x \quad (1)$$

Clearly $\sum_{k \in \mathcal{R}} A_k B^k x \in \text{span } F$. By assumption, $ABx \in \text{span } F$, and so $\sum_{k \in \mathcal{R}^c} A_k B^k x \in \text{span } F$. Now since $x \in F$ was arbitrary and $A_k \notin \text{span } F$ for any $k \in \mathcal{R}^c$, by Lemma 6 we must have $B^k \Lambda_i = 0$ for each $k \in \mathcal{R}^c, i \in \mathcal{I}$. But from Lemma 4 we know that either (i) $A_k = r \Lambda_i$ for some scalar r or (ii) $B^k \in \mathcal{Q}_1(\Lambda_i^T)$ or $B^k \in \mathcal{Q}_1(-\Lambda_i^T)$. Since $A_k \notin \text{span } F$ the first possibility is ruled out, and (ii) must hold. Consequently $B^k \Lambda_i = 0$ implies $B_{kl} \Lambda_i = 0$ for each l .

The above is true for each $i \in \mathcal{I}, k \in \mathcal{R}^c$. Now there are two possibilities:

1. Suppose that \mathcal{R} is empty. Then, for each $i \in \mathcal{I}$, and all k, l , $B_{kl}\Lambda_{li} = 0$. Fix some $i \in \mathcal{I}$ and some l such that $\Lambda_{li} \neq 0$; then $B_{kl} = 0$ for all k (the l th column of B is zero). By the definition of $G_{A,B}$, this means that there are no arcs incident into the row vertex u_l . Thus $G_{A,B}$ is not strongly connected.
2. Suppose that \mathcal{R} is nonempty. Given $k' \in \mathcal{R}$ we can write $A_{k'} = \sum_{i \in \mathcal{I}} q_i \Lambda_i$ for some constants q_i , so for any $k \in \mathcal{R}^c$,

$$B^k A_{k'} = \sum_{i \in \mathcal{I}} q_i B^k \Lambda_i = 0.$$

Moreover, since for each $i \in \mathcal{I}$ and each l , $B_{kl}\Lambda_{li} = 0$, we get $B_{kl}A_{lk'} = 0$. In terms of $G_{A,B}$, this means that there is no directed path of length 2 from any column vertex v_k with $k \in \mathcal{R}^c$ to a column vertex $v_{k'}$ with $k' \in \mathcal{R}$. Thus there is no directed path (of any length) of the form $v_k \cdots v_{k'}$ with $k \in \mathcal{R}^c$, $k' \in \mathcal{R}$, and $G_{A,B}$ is not strongly connected. \square

5. Examples

The examples below illustrate application of Theorem 1 and Corollary 2.

Example 1. Consider the special case where Corollary 2 is applied with $A = I$, \mathbf{B} the nonnegative matrices and $K = \mathbb{R}_{\geq 0}^n$. Since $\text{Im } I = \mathbb{R}^n$, clearly $\text{Im } I \not\subseteq \text{span } F$ for any nontrivial face F of $\mathbb{R}_{\geq 0}^n$. It is also immediate that for each $B \in \mathbf{B}$, the $n \times n$ matrix IB is K -positive. Now we show that that $G_{I,B}$ is strongly connected if and only if G_B is strongly connected. (i) Suppose G_B is strongly connected. An arc from vertex i to vertex j in G_B implies that $B_{ij} \neq 0$. But $B_{ij} = I_{ii}B_{ij}$, and since $I_{ii} = 1$ this implies that there exists a path $u_i v_i u_j$ in $G_{I,B}$. Thus a path from vertex i to vertex j in G_B implies a path from vertex u_i to vertex u_j in $G_{I,B}$. Thus strong connectedness of G_B implies a path between any two row vertices u_i and u_j of $G_{I,B}$. On the other hand since all arcs $u_i v_i$ exist in $G_{I,B}$, the path $u_i \cdots u_j$ implies the existence of paths $u_i \cdots v_j$, $v_i \cdots u_j$ and $v_i \cdots v_j$. (ii) Suppose $G_{I,B}$ is strongly connected. The path $v_i \cdots v_j$ in $G_{I,B}$ immediately implies a path from vertex i to vertex j in G_B . Thus we recover from Corollary 2 the fact that for a nonnegative matrix B , strong connectedness of G_B implies irreducibility of B .

Example 2. Let

$$A = \Lambda = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 0 \\ -1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let $\mathbf{B} = \mathcal{Q}_0(\tilde{B})$ and define

$$K = \{\Lambda z : z \in \mathbb{R}_{\geq 0}^3\}.$$

K is a CCP cone in \mathbb{R}^3 . (It is easy to show that any nonsingular $n \times n$ matrix defines a proper simplicial cone in \mathbb{R}^n in this way.) Since $A = \Lambda$ it is immediate that $\text{Im } A$ does not lie in the span of any nontrivial face of K . Consider any $B \in \mathbf{B}$, i.e., any matrix of the form

$$B = \begin{pmatrix} -a & 0 & b \\ 0 & c & 0 \\ d & 0 & 0 \end{pmatrix}$$

where $a, b, c, d \geq 0$. Since $A = \Lambda$,

$$AB\Lambda + (a + b + 2c + d)\Lambda = \Lambda(B\Lambda + (a + b + 2c + d)I)$$

and it can be checked that $B\Lambda + (a + b + 2c + d)I$ is nonnegative. Thus AB is K -quasipositive for all $B \in \mathbf{B}$. On the other hand $G_{A,B}$ is illustrated in Figure 1 for any $B \in \text{relint } \mathbf{B}$ and can be seen to be strongly connected. So AB is K -irreducible for any $B \in \text{relint } \mathbf{B}$.

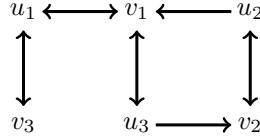


Figure 1: $G_{A,B}$ for the system in Example 2 with $B \in \text{relint } \mathbf{B}$. By inspection the digraph is strongly connected.

Example 3. We consider cases where \mathbf{B} is not the closure of a qualitative class, but is a smaller set. Suppose Λ is an $n \times m$ matrix, $A = \Lambda$ and $K = \{\Lambda z : z \in \mathbb{R}_{\geq 0}^m\}$ is a CCP cone in \mathbb{R}^n . As in the previous example, it is immediate that $\text{Im } A$ does not lie in the span of any nontrivial face of K . For some $m \times n$ matrix \tilde{B} , define

$$\mathbf{B} = \{D\tilde{B} : D \text{ is a nonnegative diagonal matrix}\}.$$

Clearly \mathbf{B} is row-complete. Suppose, further, that $\tilde{B}\Lambda$ is nonnegative, and consequently $D\tilde{B}\Lambda$ is nonnegative for any nonnegative diagonal matrix D . Then given any $B = D\tilde{B} \in \mathbf{B}$, $AB\Lambda = \Lambda(D\tilde{B}\Lambda)$, and so AB is K -positive. Corollary 2 applies and can be used to deduce K -irreducibility of AB . For example, choose

$$A = \Lambda = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 4 \\ 1 & 2 & -3 \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 3 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

As in Example 2, Since Λ is nonsingular, the cone K generated by Λ is a proper cone in \mathbb{R}^3 . It can be checked that $\tilde{B}\Lambda$ is a nonnegative matrix, so AB is K -positive for each $B \in \mathbf{B}$ (this is not the case for every $B \in \mathcal{Q}_0(\tilde{B})$). So Corollary 2 applies. $G_{A,B}$ is illustrated in Figure 2 for any $B \in \text{relint } \mathbf{B}$ (i.e.,

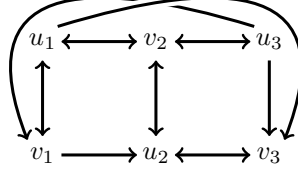


Figure 2: $G_{A,B}$ for the system in Example 3 with $B \in \text{relint } \mathbf{B}$. The digraph contains the Hamiltonian cycle $(u_1 v_3 u_2 v_2 u_3 v_1)$ and so is strongly connected.

any $B = D\tilde{B}$ where D has positive diagonal entries). The digraph is strongly connected and so, by Corollary 2, AB is K -irreducible.

Example 4. The following is an example with a cone which is not solid. Let

$$A = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 2 & 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Let $\mathbf{B} = \mathcal{Q}_0(\tilde{B})$ and define

$$K = \{\Lambda z : z \in \mathbb{R}_{\geq 0}^2\}.$$

K is a CCP cone of dimension 2 in \mathbb{R}^3 . Clearly $\text{Im } A$ does not lie in the span of any nontrivial face (i.e., any extremal) of K . Defining

$$T = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

note that $A = \Lambda T$. Consider any $B \in \mathbf{B}$, i.e., any matrix of the form

$$B = \begin{pmatrix} a & b & -c \\ d & 0 & -e \\ 0 & f & -g \end{pmatrix}$$

where $a, b, c, d, e, f, g \geq 0$. Then

$$AB\Lambda + (a + b + c + d + e + f + g)\Lambda = \Lambda(TB\Lambda + (a + b + c + d + e + f + g)I)$$

where I is the 2×2 identity matrix. It can be checked that $TB\Lambda + (a + b + c + d + e + f + g)I$ is nonnegative. Thus AB is K -quasipositive for all $B \in \mathbf{B}$. $G_{A,B}$ is illustrated in Figure 3 for any $B \in \text{relint } \mathbf{B}$ and can be seen to be strongly connected. So AB is K -irreducible for any $B \in \text{relint } \mathbf{B}$.

Example 5. As a final, nontrivial, example, let

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 \end{pmatrix}, \quad (2)$$

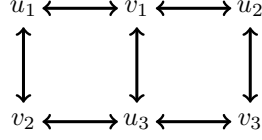


Figure 3: $G_{A,B}$ for the system in Example 4 with $B \in \text{relint } \mathbf{B}$. By inspection the digraph is strongly connected.

and $\mathbf{B} = \mathcal{Q}_0(-A^T)$. Define

$$K = \{\Lambda z : z \in \mathbb{R}_{\geq 0}^8\}.$$

Various facts can be confirmed either theoretically or via computation:

1. K is a proper cone in \mathbb{R}^4 .
2. $\text{Im } A \not\subseteq \text{span } F$ for any nontrivial face F of K .
3. For each $B \in \mathbf{B}$, AB is K -quasipositive.

Some insight into the structure of K and the proof of these facts is provided in the Appendix. It now follows from Theorem 1 that whenever $G_{A,B}$ is strongly connected, AB (and hence $AB + \alpha I$ for each $\alpha \in \mathbb{R}$) is also K -irreducible. For example, it can easily be checked that for $B \in \text{relint } \mathbf{B}$ (namely $B \in \mathcal{Q}(-A^T)$), $G_{A,B}$ is strongly connected and so AB is K -irreducible. The condition that $B \in \text{relint } \mathbf{B}$ can be relaxed while maintaining K -irreducibility. The digraphs $G_{A,B}$ for two choices of $B \in \mathbf{B}$ are illustrated in Figure 4.

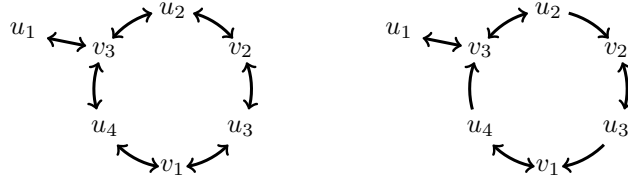


Figure 4: *Left.* The digraph $G_{A,B}$ where A is as shown in (2) and B is any matrix in $\mathcal{Q}(-A^T)$. u_i is the row vertex corresponding to row i in A , while v_i corresponds to column i . *Right.* Setting $B_{22} = B_{13} = B_{34} = 0$ removes the arcs v_2u_2 , v_1u_3 , v_3u_4 from $G_{A,B}$, but still gives a strongly connected digraph.

Confirming K -irreducibility for choices of B without the aid of Theorem 1 is possible but tedious, requiring computation of the action of AB on each of the 26 nontrivial faces of K .

6. Concluding remarks

Rather different graph-theoretic approaches to questions of irreducibility of matrices are taken in [2, 3, 4]. In [4], for example, polyhedral cones with n_K extremals were considered, and digraphs on n_K vertices constructed. Results

were presented deriving K -irreducibility of matrices from K -quasipositivity of these matrices and strong connectedness of the digraphs. The construction relies, however on knowledge of the matrix action on each extremal of K .

In our approach described here, K is not necessarily polyhedral and no knowledge of the facial structure or particular action of matrices on extremals of K is required. In compensation, however, we assume that a set of matrices with a particular structure (row-completeness) are *all* K -quasipositive, and K -quasipositivity of this entire set is essential for the proofs. This stronger assumption about K -quasipositivity allows weaker assumptions about the structure of K and the action of the matrices on faces of K . Thus both the construction of the digraph here and the assumptions are somewhat different from earlier work in this area.

Appendix A. Some details connected with Example 5

That K is closed and convex is immediate from the definition. Note that $\Lambda P = I$ where P is the nonnegative matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently K contains $\mathbb{R}_{\geq 0}^4$ and K is solid. Defining $p = (2, 1, 1, 1)^T$, $\Lambda^T p$ is strictly positive, so $p^T \Lambda z > 0$ for any nonnegative and nonzero z . Thus $p \in \text{int } K^*$, where K^* is the dual cone to K . Since K^* has nonempty interior this implies that K is pointed (if K contains a nonzero $y \in \mathbb{R}^4$ such that $y, -y \in K$, we get the contradiction $p^T y > 0$ and $p^T (-y) > 0$). Putting together these observations, K is a proper cone in \mathbb{R}^4 .

It can be checked that each Λ_i spans a different extremal of K , namely, no Λ_i can be constructed as a nonnegative combination of others. Further, the two dimensional faces of K are spanned by pairs of Λ_i for i belonging to:

$$\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}, \{5, 6\}, \{5, 7\}, \{6, 8\}, \{7, 8\}.$$

while the three dimensional faces of K are spanned by sets of four Λ_i for i belonging to:

$$\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \{3, 4, 7, 8\}, \{1, 3, 5, 7\}, \{2, 4, 6, 8\} \text{ and } \{5, 6, 7, 8\}.$$

That $\text{Im } A$ does not lie in $\text{span } F$ for any nontrivial face F can be confirmed theoretically, or checked by demonstrating for each three dimensional face F some vector z such that $Az \notin \text{span } F$. This is left to the reader.

That AB is K -quasipositive for all $B \in \mathcal{Q}_0(-A^T)$ can easily be checked. Each $B \in \mathcal{Q}_0(-A^T)$ has the form

$$B = \begin{pmatrix} a & -b & 0 & -c \\ 0 & d & -e & 0 \\ 0 & 0 & f & -g \end{pmatrix}$$

where $a, b, c, d, e, f, g \geq 0$. Defining the nonnegative matrix

$$Q = \begin{pmatrix} a+b+c & g & d & 0 & a+b+c & 0 & 0 & 0 \\ +d+g & & & & & & & \\ f & a+b+d & 0 & e+d & 0 & a+b & 0 & 0 \\ +e+f & & & & & & & \\ e & 0 & a+c+e & f+g & 0 & 0 & a+c & 0 \\ & & +f+g & & & & & \\ 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & f+g & c+e & 0 & e \\ & & & & +f+g & & & \\ 0 & 0 & b & 0 & d+e & 0 & b+d & f \\ & & & & +e+f & & & \\ 0 & 0 & 0 & b+c & 0 & d & g & b+c \\ & & & & & & & +d+g \end{pmatrix}$$

we can confirm that

$$AB\Lambda + (a + b + c + d + e + f + g)\Lambda = \Lambda Q.$$

In other words AB is K -quasipositive.

References

- [1] A. Berman and R. Plemmons. *Nonnegative matrices in the mathematical sciences*. Academic Press, New York, 1979.
- [2] G. Barker and B. Tam. Graphs for cone preserving maps. *Linear Algebra Appl.*, 37:199–204, 1981.
- [3] G. Barker and B. Tam. Graphs and irreducible cone preserving maps. *Linear and Multilinear Algebra*, 31:19–25, 1992.
- [4] H. Kunze and D. Siegel. A graph theoretic approach to strong monotonicity with respect to polyhedral cones. *Positivity*, 6:95–113, 2002.
- [5] G. Barker. The lattices of faces of a finite dimensional cone. *Linear Algebra Appl.*, 7:71–82, 1973.
- [6] J. S. Vandergraft. Spectral properties of matrices which have invariant cones. *SIAM J. Appl. Math.*, 16(6):1208–1222, 1968.
- [7] H. Smith. *Monotone Dynamical Systems: An introduction to the theory of competitive and cooperative systems*. American Mathematical Society, 1995.

- [8] M.W. Hirsch and H. Smith. *Handbook of Differential Equations: Ordinary Differential Equations, Vol II*, chapter Monotone Dynamical Systems, pages 239–357. Elsevier B. V., Amsterdam, 2005.
- [9] R. A. Brualdi and B. L. Shader. *Matrices of sign-solvable linear systems*. Number 116 in Cambridge tracts in mathematics. Cambridge University Press, 1995.
- [10] M. Banaji and C. Rutherford. P -matrices and signed digraphs. *Discrete Math.*, 311(4):295–301, 2011.
- [11] M.W. Hirsch. Systems of differential equations that are competitive or cooperative II: convergence almost everywhere. *SIAM J. Math. Anal.*, 16(3):423–439, 1985.
- [12] M. Banaji. Monotonicity in chemical reaction systems. *Dyn Syst*, 24(1):1–30, 2009.